Evaluating Student Expository Writing in Mathematics

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A previously reported coding scheme for describing student expository writing in mathematics has been developed into a set of descriptors aimed at reflecting the level of understanding being demonstrated in the writing. The original coding scheme provided a detailed analysis of the content of the writing to enable the application of the descriptors. The descriptors are intended to give researchers and teachers indications of the content of writing tasks which may help advance students' thinking. Examination of writing examples mainly from year 8 students has shown that it demonstrates little conceptual understanding of mathematics.

The use of writing as a learning aid in mathematics classes at all levels of education has received considerable attention in the mathematics education literature over the past fifteen years. The term "writing" here refers to tasks which require the use of prose as well as the usual symbolic and diagrammatic representations. The types of tasks which have been reported in use in mathematics classes vary greatly and may include journal writing (Borasi & Rose, 1989) and expository writing (Miller, 1990). A range of benefits attributed to the use of writing tasks in mathematics learning has been reported. The improved dialogue between students and their teacher and the exposure of misconceptions that would otherwise have gone undetected for longer have been clearly demonstrated.

Borasi and Rose (1989) developed a comprehensive analysis of the benefits of journal writing in mathematics and noted an increase in the content knowledge of the students as well as some exposure of their views about mathematics and its learning. In most of the reported studies, there has been little attempt to establish whether students were actually developing their understanding of mathematics through the use of writing. Many have claimed improved learning and understanding as a benefit of writing without any compelling evidence to support the claims (Powell & Lopez, 1989). Such studies have tended to assume that increased understanding will occur based on the finding from more general "writing-to-learn" studies. One study reported in Waywood (1992) and Clarke, Waywood and Stephens (1993) has provided a useful scheme for examining the development of student thinking demonstrated in their writing and this will be discussed in detail below. The study reported here extends a previous study (Shield, 1995) in which a scheme to describe student expository mathematical writing was developed. It builds on the previous study and the work of Waywood to further refine the evaluation of student expository writing with particular emphasis on the understanding of mathematics being demonstrated in the writing.

Writing and the Development of Understanding

One of the main benefits that could be realised from the use of writing activities in mathematics learning is an increase in student understanding of the subject. Understanding in mathematics learning is generally accepted to involve knowing the concepts and principles related to the procedures being used and making meaningful connections between prior knowledge and the knowledge units being learnt (Baroody & Ginsburg, 1990). A network of internal representations is established and this is developed in some way from the external representations which the learner experiences (Hiebert & Carpenter, 1992). Skemp (1976) had earlier described such understanding as "relational" and contrasted this with what he termed "instrumental" understanding which is characterised by a knowledge of many discrete mathematical processes which can be applied in very limited situations.

As well as demonstrating the type of knowledge held by the learner, the process of writing may also stimulate the development of further connections within that learner's knowledge structure. The process of connecting new ideas with existing knowledge has been described by the term "elaboration" (Hamilton, 1989) and has been studied as part of the process of reading. Elaborative processing by learners appears to make new material more meaningful and allows for more effective integration of the ideas with existing knowledge. Studies (e.g., Reder, 1980; Hamilton, 1990) have shown how elaborative processing during the learning of new ideas enhances the ability of the learner to apply those ideas in novel problems, the resulting richer knowledge base providing more options for the generation of ideas during problem solving. The process of writing can involve some of the same elaborative thinking as used in reading for students are stimulated to make meaning of the ideas they are presenting (Rose, 1989). The study reported here made use of the concept of elaboration in developing a scheme for evaluating the thinking and understanding of students as displayed in their expository writing.

Mathematical Thinking Shown in Student Writing

In a previously reported study (Shield, 1995), a coding scheme for describing the content of student expository writing in mathematics was discussed. An application of this scheme was also reported in Shield and Swinson (1994). It enables the detailed description of a piece of writing in terms of the elaborations of each main idea (known as a "kernel") and is based on Leinhardt's (1987) features of an explanation and van Dormolen's (1985) terminology for the description of the content of a mathematics textbook. The coding scheme is outlined as follows.

The van Dormolen terminology is used to describe each statement in a presentation in terms of the "aspect of mathematics" being expressed and the "level of language" used.

Aspect of mathematics: theoretical - theorems, definitions, generalisations; algorithmic - explicit 'how to do' methods; logical - the way we are allowed to handle the theory; methodological - heuristic 'how to do rules'; communicative - conventions, how to name a diagram, write a proof.

Level of language: exemplary - demonstrative, related to a specific example; relative - generalised, not related to a specific example. Within each level, the language may be procedural or descriptive.

The method then involves partitioning the writing into separate elaborations of the kernel. Separate elaboration can often be distinguished by changes in the aspect of mathematics being expressed or the level of language being used. The elaborations which may be found in student expository writing are:

Kernel	definition or general statement of the procedure
Goal statement	identification of the concept or procedure being explained
Demonstration	a worked example of the concept or procedure which may be elaborated with: (a) symbolic representation; (b) verbal description; (c) diagrammatic representation; (d) statement of convention
Legitimisation	justification for the procedure or part of it using known principles

Link to prior knowledge	extensions of prior knowledge, reference to everyday experience
Practice exercises	set of questions to be answered by the reader by modelling on the demonstration

Not all of the elaborations are usually included in individual pieces of writing and it has been found that legitimisation was rarely present (Shield, 1995). A rich description of the writing can be developed and some indication of the level of understanding may be inferred. That is, the number of elaborations used indicates how much the student has linked the idea or procedure with various representations and ideas from prior knowledge. The level of language and aspect of mathematics in some of the statements are also indicative of understanding. In the earlier study (Shield, 1995), it was found that many students rarely expressed anything other than the algorithmic aspect of mathematics and only approximately half the students could describe a procedure in generalised (relative) language, the remainder always writing only in terms of a specific example.

Establishing the Level of Understanding Evident in Writing

The present study was designed to extend the coding scheme to provide a way of more systematically determining the level of understanding being displayed in the writing. Methods reported in two earlier papers assisted in this development. In one study (Waywood, 1992) a scheme to assess journal writing based on a set of descriptors was developed. In that study, journal writing included the tasks summarising, collecting examples, questioning and discussing. Each of these tasks was further delineated into four sub-categories which enabled a detailed description of the processes evidenced in each student's writing. In conjunction with the categories, a set of progress descriptors was described. From the pattern established, a global progress categorisation of the student's journal writing into one of three modes, namely recount, summary, or dialogue could be established. Parts of the definitions of these modes are reproduced below.

> RECOUNT When students are writing in this mode, they interpret the tasks in terms of concrete things to be done: to write a summary means *record*: . . .
> SUMMARY When students are writing in this mode they interpret the tasks as requiring involvement. The involvement is utilitarian. Describing gives way to *stating* and *organising*. . . . Journals show students trying to form an overview.
> DIALOGUE When writing in this mode, students see the task as requiring them to generate mathematics. . . . Summaries are about *integrating*; questions are about *analysing* and *directing*; examples are *paradigms*; and discussing is about *formulating* arguments. . .(p. 38)

With further analysis of the students' journals, and also analysis of questionnaires for students and teachers developed for the purpose, Clarke, Waywood and Stephens (1993) confirmed and further described these progress modes and formulated a detailed description of student thinking associated with each. They were able to establish a general trend that as experience with this type of journal writing increased, the students' writing tended to progress through the modes.

Shepard (1993) addressed the issue of the role of writing in conceptual development from a theoretical standpoint. He adapted the work of a number of cognitive psychologists to describe learning in three phases, namely initial, intermediate and terminal. Within each phase, writing categories were described at progressive levels. The descriptions are summarised as follows:

Initial	record	transcription
	report	summaries - no inferences, just information

	generalised narrative	personal examples of concepts, ideas explained in their own words
Intermediate	low-level analogic	explains how to solve problem, explains what is wrong with incorrect working
	analogic	explain how concepts are related, explain how and why procedures apply or don't apply
Terminal	analogic-tautologic	propose alternative ways and understand how they are different
	tautologic	produce new method for a certain type of problem

These descriptions in some ways parallel those developed by Waywood (1992). Shepard's aim was to provide teachers and researchers with guidance for designing writing assignments in mathematics which could assist in moving students from the rote memory of algorithms towards a more abstract and integrative type of thinking.

In the present study, examples of student mathematical writing from earlier studies (Shield & Swinson, 1994; Shield, 1995) which had been coded with the earlier detailed scheme were re-examined with the systems summarised above (Waywood, 1992; Shepard, 1993). The aim was to develop, by a synthesis of the three systems, a set of descriptions for recognisable levels of elaboration of mathematical ideas which would reflect the thinking and understanding being displayed by students in their writing. Five levels of understanding have been identified, the highest corresponding reasonably closely to Shepard's (1993) intermediate/analogic mode and Waywood's (1992) summary mode. The examples used came mainly from year 8 students and no higher levels were apparent. In the following paragraphs, the five levels are described and exemplified. The levels and their examples are discussed using the terminology from the coding scheme outlined above.

1. *Approach.* At this level, the student writer presents either an unelaborated demonstration of the algorithm or a simple algorithmic description of the method which lacks sufficient definition to be a true kernel. In the first example in the appendix, the student writer was asked to explain about arranging the numbers 4831, 5107, 4970 and 5015 in ascending order. The complete response is shown. In the second example, the task was to explain about working questions such as: "What is 35% of 80?" One student's complete response is shown.

2. *Recount.* At this level, writers generally reproduce a demonstration of the concept or procedure. The demonstration is complete and may be elaborated both symbolically and verbally and the language is exemplary (related only to the specific example). It may also be elaborated diagrammatically when appropriate. The aspect of mathematics being expressed is algorithmic. In the example in the appendix, the writer was responding to the same task as in the second "approach" example. The working is complete and the algorithmic nature is emphasised by the steps.

3. *Generalise.* Writing at this level also focuses on a demonstration of the concept or procedure which may be elaborated symbolically, verbally and diagrammatically when appropriate. The main feature which distinguishes it from level 2 is the inclusion of a definition or procedural statement in relative (generalised) language, that is, a kernel is clearly stated. The aspect of mathematics being expressed is algorithmic and the presentation may include some practice exercises for the reader. In the example in the appendix, the student was asked to write a letter to an absent friend to explain all about highest common factor. The writer has stated a goal for the presentation in lines 1 to 3. Line 5 to 11 contain the relative statement of the procedure which shows that the student

has been able to generalise the method into an algorithm. In the earlier study (Shield, 1995), it was found that while most students could produce a demonstration, only about half of them could also make a general statement of the procedure.

4. *Link.* Writing at this level contains the same elaborations as at level 3 but also includes some links with prior knowledge or everyday applications of the concept or procedure. Students writing in this way are beginning to demonstrate an understanding of the relational nature of mathematics. In the example provided, the writer responding to the same task as the writer in the level 3 example has included a discussion of factors in lines 5 to 8 in preparation for the kernel and demonstration which follow.

5. *Integrate.* At this level, the student writer integrates a number of relevant ideas to arrive at an elaborated presentation which includes some statements expressing theoretical and methodological aspects of mathematics in relative language, as well as the types of content described in the previous levels. There is a logical organisation of the ideas. In the example, a year 9 student was asked to write a letter to an absent friend to explain all about solving equations, linear equations being the only ones so far covered in class work. In the example, the writer has firstly addressed prior knowledge of what an equation is and then made a general statement (lines 5 and 6) about what it means to solve an equation, a theoretical aspect of mathematics. The following two sentences (lines 6 to 10) express methodological aspects of mathematics in relative language. The remainder of the presentation then focuses on the algorithmic aspects of the procedure except for the statement on line 12 about setting out which is a communicative aspect conveying an accepted convention.

Conclusion

The scheme described in this paper provides a method of evaluating the level of understanding of specific mathematical ideas being displayed in students' expository writing. The earlier coding scheme (Shield, 1995) enables a detailed description of the content of the writing, rather than a global description, and enables decisions to be made regarding the level to be assigned. In working with this scheme, so far there has been little difficulty in assigning writing examples to one of the levels. It is expected that in further work higher levels will be demonstrated as indicated by Waywood (1992) and Shepard (1993). From the examples collected by this author and from many of the examples quoted in the literature on writing in mathematics at the school level, it is apparent that most student writing is at level 3 or lower. If the writing in some way reflects the conceptions of the mathematical ideas and beliefs held by the students, then it indicates that students are learning mathematical procedures in an instrumental rather than relational way. The aim of this scheme is to provide researchers and teachers with a framework for further developing students' thinking through the use of writing by indicating the types of content that students might be encouraged, by the use of examples, to include in their writing.

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APPENDIX

Level 1: APPROACH

You look at the lowest number and to the next lowest and so on until you have the highest number last.

~~sorting~~

You can use a calculator and put in $35 \div 100 \times 80$ then it will give a Aswer.

Level 2: RECOUNT

you devide the 35% by 100 because (% = 100)

$35 \div 100 = 0{\cdot}35$

multiply 0·35 by 80 because that is what you want to know

$0{\cdot}35 \times 80 = 28\%$

STEP one $35 \div 100 = 0{\cdot}35$

STEP TWO - $0{\cdot}35 \times 80 = 28$

STEP THREE = Answer = 28%

Level 3: GENERALISE

Dear Jane Doe,

I am writing to tell you about how to find a Highest Common Factor (HCF) and the Lowest Common Mutiple (LCM) of 2 numbers.

H.C.F. To do this you must have 2 numbers. E.g. 16 and 40. You must then start writing down their factors. When you finish doing this you must find the highest number that occur in both numbers. That number is you H.C.F.!!

e.g. 16 → 1, 2, 4, (8), 16

40 → 1, 2, 4, 5, (8), 10, 20, 40

∴ the H.C.F is 8 because it is the largest number that occurs in both numbers.

Level 4: LINK

Dear Jessy,

Well I know how you've been away for a couple of weeks and Mrs gave me the job to explain about Highest Common Factor and Lowest Common Multiple. Well let me start with HCF. A factor is small than the number that you start off with. The factors are numbers that can divide into the number evenly for eg 16=1,2,4,8,16. Do you see. HCF is highest common factor, so you take two numbers 12 and 16 and you list the factors. When you've done that you look and see the highest factor both of them have otherwise you always have one.

eg 12=1,2,3,(4),6,12
16=1,2,(4),8,16

They both have 4 so you're answer is 4 Get it.

Level 5: INTEGRATE

Solving Equasions.

In an equasion it shows that something is equal to something else: $5+6 = 7+4$

Sometimes in an equasion their is an unknown number (x):

$3x + 2 = 11$

To find the unknown quantity is called solving the equasion. When solving an equasion the idea is to get x on its own. To do this you have to cancel out (do the [illegible]) all the other numbers on that paticular side of the equasion. Throughout the sum it is essential that the equasion is balanced at all times. To do this ~~[illegible]~~ you have to make sure whatever you do to one side you do to the other. The equasion has to be set out as follows.

For ~~example~~ $3x + 2-2 = 11-2$ (You have to take away the 2 ~~[illegible]~~ from both ~~sides~~)

$3x \div 3 = 9 \div 3$ (You have to divide both sides by 3 to leave x

$x = 3$ on its own reading the answer in the equasion)

You can check the answer by substituting you answer with x and if the ~~[illegible]~~ equasion is equal you are right eg.

$3 \times 3 + 2 = 11$

$9 + 2 = 11$

$11 = 11$

Try these to see how you go (Remember you can check your answer).

1. $5x - 20 = 30$
2. $x - 14 = {}^{-}2$
3. $3 - 2x = {}^{-}1$